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Effects of anti-symmetric nonlinear viscous damping on the force transmissibility of multi-degree of freedom structures

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Abstract In the present study, the concept of the output frequency response function is applied to theoretically investigate the force transmissibility of multi-degree of freedom (MDOF) structures with a nonlinear anti-symmetric viscous damping. The results reveal that an anti-symmetric nonlinear viscous damping can significantly reduce the transmissibility over all resonance regions for MDOF structures while it has almost no effect on the transmissibility over non-resonant and isolation regions. The results indicate that the vibration isolators with an anti-symmetric damping characteristic have great potential to overcome the dilemma in the design of linear viscously damped vibration isolators where an increase of the damping level reduces the force transmissibility over resonant region but increases the transmissibility over non-resonant regions. © *2011 The Chinese Society of Theoretical and Applied Mechanics.* [doi:10.1063/2.1106304]

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Vibration isolation is considered to be an effective method to reduce the vibration energy transmission so as to protect equipment or structures from vibration disturbances.[1] Transmissibility is a concept used to describe the effectiveness of a vibration isolation system. Force transmissibility is defined in the frequency domain as the ratio of the force output of a vibration isolation system to the force input at an operating frequency of concern. Under a linearity assumption, the performance characteristics of vibration isolators have been widely reported.[1] The transmissibility in such cases can be explicitly expressed as a simple function of various factors that can be used for the design. A comprehensive review can be found in Ref. 2. In spite of these significant achievements in the design of vibration isolation systems, there is a well-known dilemma[3] associated with the design of viscously damped linear vibratrion isolators, that is, when the damping level is increased to reduce the transmissibility over the resonant frequency regions, the transmissibility is increased over the other regions of frequencies where a desired vibration isolation is often required.

Recently, nonlinear vibration isolators have received more and more attention from researchers because all practical systems are inherently nonlinear, and taking the effects of nonlinearity into account in designs can achieve better performance. A very comprehensive survey about recent developments of nonlinear vibration isolators is contributed by Ibrahim,[4] which it was revealed that the introduction of nonlinear damping and stiffness are really of great benefit in vibration isolation. Generally speaking, the design of nonlinear vibration isolation systems is a complicated and difficult challenge. This is mainly due to the difficulties with the analysis of nonlinear systems, since a closed-form analytic solution to nonlinear differential equations is possible only for some special classes of nonlinear differential equations.[5] As a result, researchers often have to simplify the analysis by resorting to single degree of freedom (SDOF) or low dimensional nonlinear models. However, even with simplified models, the analysis is still not an easy task. For the study of nonlinear vibration isolation systems, for example, an immediate difficulty is that it is hard to derive an explicit analytical description for the relationship between the system nonlinear characteristic parameters and the force transmissibility.

Most recently, by applying the concept of the output frequency response functions (OFRFs),[6,7] the authors[8] have revealed that, for SDOF vibration isolators, a cubic nonlinear viscous damping characteristic can produce an ideal vibration isolation such that only transmissibility over the resonant frequency region is modified by the nonlinear damping effect but the transmissibility over non-resonant frequency regions remain almost unaffected. Therefore, by introducing a cubic nonlinear viscous damping to SDOF vibration isolators, the dilemma associated with the design of linear viscously damped vibration isolators can be overcome. In this paper, we present the efforts of extending the analysis result to general multi-degree of freedom (MDOF) structures with an anti-symmetric nonlinear viscous damping device. This not only extends the significant theoretical conclusion reached in Ref. 8 to a much more general case, but also provides an important foundation for the development of novel passive solutions to more complicated vibration isolation problems.

Consider MDOF structures with an anti-symmetric nonlinear viscous damping characteristic located be-

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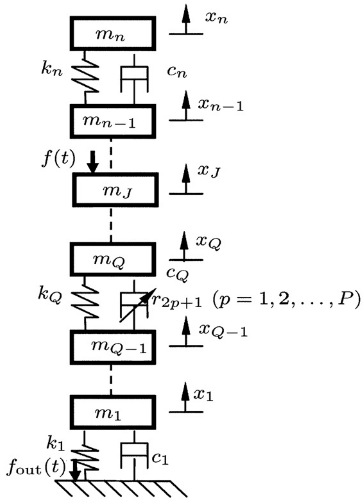

Fig. 1. The MDOF structure with an anti-symmetric nonlinear damping characteristic.

tween the $(Q-1)$th and Qth masses as shown in Fig. 1, where

$$f(t) = A\sin(\Omega t). \tag{1}$$

is the harmonic force acting on the Jth mass with frequency Ω and magnitude A, $f_{\rm out}(t)$ is the force transmitted to the supporting base, and $x_i(t)$ is the displacement of mass i ($i = 1, 2, \ldots, n$). The restoring damping force of the nonlinear damper located between the $(Q-1)$th and Qth masses is described by

$$F_{\rm Non} = \sum_{i=1}^{P} r_{(2i+1)}(\dot{x}_{Q-1} - \dot{x}_Q)^{2i+1}, \tag{2}$$

where $r_{(2i+1)}$ ($i = 1, 2, \ldots, P$) are the parameters of the anti-symmetric nonlinear viscous damping characteristic. The motion governing equations of the MDOF structure can be written in the following matrix form

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{C}\dot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{F}_N = \boldsymbol{F}(t), \tag{3}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the system mass, damping and stiffness matrix, respectively, and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)'$, $\boldsymbol{F}(t) = (\overbrace{0, ..., 0}^{J-1}, f(t), 0, ..., 0)'$, and

$$\boldsymbol{F}_N = \left(\overbrace{0, \cdots, 0}^{Q-2},\ F_{\rm Non},\ -F_{\rm Non},\ \overbrace{0, \cdots, 0}^{n-Q}\right)'$$

Damping matrix $\boldsymbol{C}$ is assumed to be proportional to stiffness matrix $\boldsymbol{K}$, e.g., $\boldsymbol{C} = \mu\boldsymbol{K}$. The force transmitted to the supporting base $f_{\rm out}(t)$ can be evaluated as follows

$$f_{\rm out}(t) = k_1 x_1 + c_1\dot{x}_1. \tag{4}$$

Denote $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)'$ and $\boldsymbol{x} = \boldsymbol{\Phi}\boldsymbol{y}$ where

$$\boldsymbol{\Phi} = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \cdots & \Phi_{1n} \\ \Phi_{21} & \Phi_{22} & \cdots & \Phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{n1} & \Phi_{n2} & \cdots & \Phi_{nn} \end{pmatrix} \tag{5}$$

is the mode shape matrix[9] which is generated by solving the following eigenvalue problem

$$\left(\boldsymbol{K} - \bar{\omega}^2\boldsymbol{M}\right)\boldsymbol{\Phi} = 0, \tag{6}$$

where $\bar{\omega}$ denotes the eigenvalue of the system. Multiplying Eq. (3) by $\boldsymbol{\Phi}^{\rm T}$ and then replacing $\boldsymbol{x}$ with $\boldsymbol{\Phi}\boldsymbol{y}$ yields

$$\begin{aligned} &\boldsymbol{\Phi}^{\rm T}\boldsymbol{M}\boldsymbol{\Phi}\ddot{\boldsymbol{y}} + \boldsymbol{\Phi}^{\rm T}\boldsymbol{C}\boldsymbol{\Phi}\dot{\boldsymbol{y}} + \boldsymbol{\Phi}^{\rm T}\boldsymbol{K}\boldsymbol{\Phi}\boldsymbol{y} + \\ &\quad \boldsymbol{\Phi}^{\rm T}\boldsymbol{F}_N = \boldsymbol{\Phi}^{\rm T}\boldsymbol{F}(t). \end{aligned} \tag{7}$$

Moreover, it is known that $\boldsymbol{\Phi}^{\rm T}\boldsymbol{M}\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^{\rm T}\boldsymbol{K}\boldsymbol{\Phi}$ can be expressed in a more succinct form

$$\overline{\boldsymbol{M}} = \boldsymbol{\Phi}^{\rm T}\boldsymbol{M}\boldsymbol{\Phi} = \begin{pmatrix} \bar{m}_1 & 0 & \cdots & 0 \\ 0 & \bar{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{m}_n \end{pmatrix},$$

$$\overline{\boldsymbol{K}} = \boldsymbol{\Phi}^{\rm T}\boldsymbol{K}\boldsymbol{\Phi} = \begin{pmatrix} \bar{k}_1 & 0 & \cdots & 0 \\ 0 & \bar{k}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{k}_n \end{pmatrix}.$$

Matrix $\overline{\boldsymbol{M}}$ is referred to as the modal mass matrix, and $\overline{\boldsymbol{K}}$ the modal stiffness matrix. Therefore, Eq. (3) can be decomposed into a series of equations as follows

$$\begin{aligned} &\ddot{y}_i + \bar{\mu}_i\bar{\omega}_i\dot{y} + \bar{\omega}_i^2 y_i + \\ &\qquad \left(\bar{\Phi}_{(Q-1)i} - \bar{\Phi}_{Qi}\right)\sum_{i=1}^{P} r_{(2i+1)}\dot{v}^{2i+1} = \\ &\qquad \bar{\Phi}_{Ji} A\sin(\omega t), \\ &x_i = \left(\Phi_{i1}, \Phi_{i2}\cdots, \Phi_{in}\right) y, \\ &v = x_{Q-1} - x_Q, \\ &f_{\rm out} = c_1\dot{x}_1 + k_1 x_1, \\ &(i = 1, 2, \ldots, n), \end{aligned} \tag{8}$$

where $\bar{\mu}_i = \mu\sqrt{\bar{k}_i/\bar{m}_i}$, $\bar{\omega}_i^2 = \bar{k}_i/\bar{m}_i$, $\bar{\Phi}_{1i} = \Phi_{1i}/\bar{m}_i$, and $\bar{\Phi}_{Ji} = \Phi_{Ji}/\bar{m}_i$.

Based on the Volterra series theory of nonlinear systems, the relationships between the output $x_i(t)$ or its alternative representation $y_i(t)$ ($i = 1, 2, \cdots, n$) and the input force $f(t)$ of Eq. (3) or (8) can be expressed by using generalised frequency response function (GFRF),[10] which can be recursively determined by using the algorithm in Billings and Peyton Jones[11] as follows

$$\begin{aligned} H_{y_i}^{(1)}(\omega_1) &= \frac{-\bar{\Phi}_{Ji}}{L_{y_i}(\omega_1)}, \\ H_{x_i}^{(1)}(\omega_1) &= \sum_{d=1}^{n}\frac{-\bar{\Phi}_{Jd}\Phi_{id}}{L_{y_d}(\omega_1)}, \\ H_{v}^{(1)}(\omega_1) &= H_{x_{(Q-1)}}^{(1)}(\omega_1) - H_{x_Q}^{(1)}(\omega_1) \\ &= \sum_{d=1}^{n}\frac{-\bar{\Phi}_{Jd}(\Phi_{(Q-1)d} - \Phi_{Qd})}{L_{y_d}(\omega_1)}, \\ H_{f_{\rm out}}^{(1)}(\omega_1) &= (jc_1\omega_1 + k_1)H_{x_i}^{(1)}(\omega_1) \end{aligned}$$

$$= (jc_1\omega_1 + k_1)\sum_{d=1}^{n}\frac{-\bar{\varPhi}_{Jd}\varPhi_{1d}}{L_{y_d}(\omega_1)}, \tag{9}$$

and

$$\begin{aligned}
&H_{y_i}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\frac{\bar{\varPhi}_{(Q-1)i}-\bar{\varPhi}_{Qi}}{L_{y_i}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})}\cdot \\
&\quad\varLambda_v^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}), \\
&H_{y_i}^{(2L)}(\omega_1,\omega_2,...,\omega_{2L}) = 0, \\
&H_{x_i}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\ell_{x_i}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})\cdot \\
&\quad\varLambda_v^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}), \\
&H_{x_i}^{(2L)}(\omega_1,\omega_2,...,\omega_{2L}) = 0, \\
&H_{v}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\ell_{v}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})\cdot \\
&\quad\varLambda_v^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}), \\
&H_{v}^{(2L)}(\omega_1,\omega_2,...,\omega_{2L}) = 0, \\
&H_{f_{\text{out}}}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\ell_{f_{\text{out}}}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})\cdot \\
&\quad\varLambda_v^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}), \\
&H_{f_{\text{out}}}^{(2L)}(\omega_1,\omega_2,...,\omega_{2L}) = 0, \\
&(i=1,2,\cdots,n)\ \text{ and }\ (L=2,3,\cdots,N/2),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
&L_{y_i}(\omega_1+\omega_2+\cdots+\omega_p) = \\
&\quad -[-(\omega_1+\omega_2+\cdots+\omega_p)^2 + \\
&\quad j\bar{\mu}_i\bar{\omega}_i(\omega_1+\omega_2+\cdots+\omega_p)+\bar{\omega}_i^2],
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\ell_{x_i}(\omega_1+\omega_2+\cdots+\omega_p) = \\
&\quad\sum_{d=1}^{n}\frac{\varPhi_{id}\left(\bar{\varPhi}_{(Q-1)d}-\bar{\varPhi}_{Qd}\right)}{L_{y_d}(\omega_1+\omega_2+\cdots+\omega_p)},
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\ell_{v}(\omega_1+\omega_2+\cdots+\omega_p) = \\
&\quad\sum_{i=1}^{n}\frac{\left(\bar{\varPhi}_{(Q-1)i}-\bar{\varPhi}_{Qi}\right)^2}{L_{y_i}(\omega_1+\omega_2+\cdots+\omega_p)},
\end{aligned} \tag{13}$$

$$\begin{aligned}
&\ell_{f_{\text{out}}}(\omega_1+\omega_2+\cdots+\omega_p) = \\
&\quad[jc_1(\omega_1+\omega_2+\cdots+\omega_p)+k_1]\cdot \\
&\quad\ell_{x_1}(\omega_1+\omega_2+\cdots+\omega_p),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\varLambda_v^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\sum_{i=1}^{P} r_{(2i+1)}H_v^{(2L-1,2i+1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}),
\end{aligned} \tag{15}$$

with

$$\begin{aligned}
H_v^{(L,p)}(.) = &\sum_{i=1}^{L-p+1} H_v^{(i)}\left(\omega_1,\omega_2,\cdots,\omega_i\right)H_v^{(L-i,p-1)}\cdot \\
&\left(\omega_{i+1},\omega_{i+2}+\cdots,\omega_L\right)\left(j\omega_1+j\omega_2+\cdots+j\omega_i\right),
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
H_v^{(L,1)}\left(\omega_1,\omega_2,\cdots,\omega_L\right) = &H_v^{(L)}\left(\omega_1,\omega_2,\cdots,\omega_L\right)\cdot \\
&\left(j\omega_1+j\omega_2+\cdots+j\omega_L\right).
\end{aligned} \tag{17}$$

Moreover, from the results recently revealed by the authors in Refs. 6, 7, the $(2L-1)$th order GFRFs ($L = 2,3,\cdots,N/2$) of Eq. (3) or (8) associated with v, y_i ($i = 1,\ 2,\ \ldots,\ n$) and f_{out}, respectively, can be determined as follows

$$\begin{aligned}
&H_v^{(2L-1)}(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}) = \\
&\quad\frac{\prod\limits_{i=1}^{2L-1}\left[j\omega_i H_v^{(1)}(j\omega_i)\right]}{\ell_v^{-1}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})}\cdot \\
&\quad\sum_{\left(j_3j_4\cdots j_{(2P+1)}\right)\in J_{(2L-1)}} r_3^{j_3}\cdots r_{(2P+1)}^{j_{(2P+1)}}\cdot \\
&\quad\varTheta_{(2L-1)}^{\left(j_3j_4\cdots j_{(2P+1)}\right)}\left(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}\right),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&H_{y_i}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\frac{\left(\bar{\varPhi}_{(Q-1)i}-\bar{\varPhi}_{Qi}\right)\prod\limits_{d=1}^{2L-1}\left[j\omega_d H_v^{(1)}(\omega_d)\right]}{L_{y_i}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})}\cdot \\
&\quad\sum_{\left(j_3j_4\cdots,j_{(2P+1)}\right)\in J_{(2L-1)}} r_3^{j_3}\cdots r_{(2P+1)}^{j_{(2P+1)}}\cdot \\
&\quad\varTheta_{(2L-1)}^{\left(j_3j_4\cdots j_{(2P+1)}\right)}\left(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}\right),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&H_{f_{\text{out}}}^{(2L-1)}(\omega_1,\omega_2,...,\omega_{(2L-1)}) = \\
&\quad\ell_{f_{\text{out}}}(\omega_1+\omega_2+\cdots+\omega_{(2L-1)})\cdot \\
&\quad\prod_{d=1}^{2L-1}\left[j\omega_d H_v^{(1)}(\omega_d)\right]\cdot \\
&\quad\sum_{\left(j_3j_4\cdots j_{(2P+1)}\right)\in J_{(2L-1)}} r_3^{j_3}r_4^{j_4}\cdots r_{(2P+1)}^{j_{(2P+1)}}\cdot \\
&\quad\varTheta_{(2L-1)}^{\left(j_3j_4\cdots j_{(2P+1)}\right)}\left(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}\right),
\end{aligned} \tag{20}$$

where $J_{(2L-1)}$ is a set of P dimensional nonnegative integer vectors which contains the exponents of those monomials $r_3^{j_3}, r_4^{j_4}\cdots r_{(2P+1)}^{j_{(2P+1)}}$ which are present in the polynomial representations (18)-(20). The $\varTheta_{(2L-1)}^{\left(j_3j_4\cdots j_{(2P+1)}\right)}\left(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}\right)$ is a function of frequency variables $\omega_1,\omega_2,\cdots,\ \omega_{(2L-1)}$ and the system's linear characteristic parameters. The $J_{(2L-1)}$ and $\varTheta_{(2L-1)}^{\left(j_3j_4\cdots j_{(2P+1)}\right)}\left(\omega_1,\omega_2,\cdots,\omega_{(2L-1)}\right)$ in Eqs. (18)-(20) can be determined by applying the recursive algorithm given in Ref. 7, which was proposed by the authors. For example, applying the algorithm for $n = 1,\ 2,\ 3$ respectively yields

$$J_3 = \{(1,0)\},\ J_5 = \{(2,0),(0,1)\},$$

$$J_7 = \{(3,0), (1,1)\}.$$

and

$$\Theta_3^{(1,0)}(j\omega_1, j\omega_2, \cdots, j\omega_3) = 1,$$

$$\Theta_5^{(2,0)}(j\omega_1, j\omega_2, \cdots, j\omega_5) = B_3,$$

$$\Theta_5^{(0,1)}(j\omega_1, j\omega_2, \cdots, j\omega_5) = 1,$$

$$\Theta_7^{(3,0)}(j\omega_1, j\omega_2, \cdots, j\omega_7) = B_3B_3 + B_5B_3,$$

$$\Theta_7^{(1,1)}(j\omega_1, j\omega_2, \cdots, j\omega_7) = B_5 + B_3,$$

where

$$B_D = \begin{cases} 1, & \text{if } D = 1 \\ \dfrac{j\omega_{l(1)} + j\omega_{l(2)} + \cdots + j\omega_{l(D)}}{\ell_v^{-1}(\omega_{l(1)} + \omega_{l(2)} + \cdots + \omega_{l(D)})}, & \text{if } D \geq 2 \end{cases},$$

$$\omega_{l(i)}, (i = 1, 2, \cdots, D) \in \left\{\omega_1, \omega_2, \cdots, \omega_{(2L-1)}\right\}. \quad (21)$$

Therefore, if $P = 2$, the GFRFs associated with f_{out} of Eq. (3) or (8) can, for example, be determined as

$$\begin{aligned} &H_{f_{\text{out}}}^{(3)}(\omega_1, \omega_2, ..., \omega_3) = r_3 \hbar_{f_{\text{out}}}^{(3)}(\omega_1, \omega_2, ..., \omega_3), \\ &H_{f_{\text{out}}}^{(5)}(\omega_1, \omega_2, ..., \omega_5) = \\ &\quad \left(r_3^2 B_3 + r_5\right) \hbar_{f_{\text{out}}}^{(5)}(\omega_1, \omega_2, ..., \omega_5), \\ &H_{f_{\text{out}}}^{(7)}(\omega_1, \omega_2, ..., \omega_7) = \left[r_3^3 (B_3B_3 + B_5B_3) + \right. \\ &\quad \left. r_3 r_5 (B_5 + B_3)\right] \hbar_{f_{\text{out}}}^{(7)}(\omega_1, \omega_2, ..., \omega_7), \end{aligned} \quad (22)$$

where

$$\begin{aligned} &\hbar_{f_{\text{out}}}^{(2L-1)}(\omega_1, \omega_2, ..., \omega_{(2L-1)}) = \\ &\quad \ell_{f_{\text{out}}}(\omega_1 + \omega_2 + \cdots + \omega_{(2L-1)}) \cdot \\ &\quad \sum_{d=1}^{n} \frac{\Phi_{1d}\left(\bar{\Phi}_{(Q-1)d} - \bar{\Phi}_{Qd}\right)}{L_{y_d}(\omega_1 + \omega_2 + \cdots + \omega_{(2L-1)})} \cdot \\ &\quad \prod_{d=1}^{2L-1} \left[j\omega_d H_v^{(1)}(\omega_d)\right]. \end{aligned} \quad (23)$$

Similarly, $\Theta_{(2L-1)}^{(j_3 j_4 \cdots j_{(2P+1)})}\left(\omega_1, \omega_2, \cdots, \omega_{(2L-1)}\right)$ in Eqs. (18)-(20) can be uniformly expressed as

$$\begin{aligned} &\Theta_{(2L-1)}^{(j_3 j_4 \cdots j_{(2Q+1)})}\left(j\omega_1, j\omega_2, \cdots, j\omega_{(2L-1)}\right) = \\ &\quad \sum_{D=1}^{\bar{n}} \prod_{i=1}^{D} B_{l(2D+1)} = \\ &\quad \sum_{D=1}^{\bar{n}} \prod_{i=1}^{D} \frac{j\omega_{l(1)} + j\omega_{l(2)} + \cdots + j\omega_{l(2D+1)}}{\left[\ell_v^{-1}(\omega_{l(1)}) + \omega_{l(2D+1)}\right]}, \end{aligned} \quad (24)$$

where $\bar{n}$ is an integer dependent on $(2L-1)$.

For SDOF vibration isolators, the concept of force transmissibility has been well defined and extensively investigated by researchers. However, the concept of force transmissibility for MDOF structures has not been well established yet. Several definitions have been proposed by different researchers.[12,13] Based on the transmissibility definition proposed by Hsueh,[12] in the present study, the force transmissibility of Eq. (3) or (8) is defined as

$$T_R(\Omega) = \left|\frac{F_{\text{out}}(\omega)}{A}\right|_{\omega=\Omega}, \quad (25)$$

where $F_{\text{out}}(\omega)$ is the spectrum of $f_{\text{out}}(t)$ given in Eq. (4). This definition of the force transmissibility can be justified by the fact that the fundamental harmonic component is usually the dominant component in the output response of Eq. (3) or (8). Obviously, when $A = 1$,

$$T_R(\Omega) = |F_{\text{out}}(\Omega)|. \quad (26)$$

The transmissibility can be studied by looking at the fundamental harmonic component of $f_{\text{out}}(t)$ — the sinusoidal force transmitted to the supporting base.

The OFRF is a concept recently proposed by the authors in Refs. 13 and 14 for the study of the output frequency response of nonlinear Volterra systems. For nonlinear Volterra systems that can equally be described by a polynomial type nonlinear differential equation model, which has been widely used for the modeling of practical physical systems, the system output spectrum can be represented by an explicit polynomial function of the model parameters which define the system nonlinearity. This result is referred to as the OFRF, which provides a significant analytical link between the output frequency response and nonlinear characteristic parameters for a wide range of practical nonlinear systems.

According to Refs. 13, 14, when $f(t) = \sin(\Omega t)$, the spectrum of the output $f_{\text{out}}(t)$ of Eq. (3) or (8) at frequency Ω, that is, the system force transmissibility $T_R(\Omega)$ can be expressed as

$$T_R(\Omega) = \Gamma_1(\Omega) + \Gamma_3(\Omega) + \sum_{L=3}^{(N+1)/2} \Gamma_{2L-1}(\Omega), \quad (27)$$

where

$$\begin{aligned} &\Gamma_{(2L-1)}(\Omega) = \frac{(2L-1)!}{2^{2L-2}(L-1)!L!} H_{f_{\text{out}}}^{(2L-1)} \cdot \\ &\quad (\overbrace{\Omega, \cdots, \Omega}^{L}, \overbrace{-\Omega, \cdots, -\Omega}^{L-1}), \\ &\quad L = 1, 2, \cdots, [(N+1)/2]. \end{aligned} \quad (28)$$

Substituting Eqs. (9), (10) and (20) into Eq. (28) yields

$$\Gamma_1(\Omega) = (k_1 + jc_1\Omega) \sum_{i=1}^{n} \frac{-\bar{\Phi}_{Ji}\Phi_{1i}}{L_{y_i}(\Omega)}, \quad (29)$$

$$\Gamma_3(\Omega) = jr_3 \frac{3\Omega^3}{4} \ell_{f_{\text{out}}}(\Omega) H_v^{(1)}(\Omega) \left|H_v^{(1)}(\Omega)\right|^2, \quad (30)$$

$$\Gamma_{(2L+1)}(\Omega) = j\frac{(2L+1)!}{2^{2L}(L+1)!L!} \Omega^{2L+1} \ell_{f_{\text{out}}}(\Omega) \cdot$$

$$H_v^{(1)}(\Omega)\left|H_v^{(1)}(\Omega)\right|^{2L}\cdot \sum_{(j_3,j_4,\cdots,j_{(2P+1)})\in J_{(2L+1)}} r_3^{j_3}r_4^{j_4}\cdots r_{(2P+1)}^{j_{(2P+1)}}\Theta_{(2L+1)}^{(j_3j_4\cdots j_{(2P+1)})}\cdot (\overbrace{\Omega,\cdots,\Omega}^{L+1},\overbrace{-\Omega,\cdots,-\Omega}^{L}),$$
$$(L=2,3,\cdots,(N-1)/2). \tag{31}$$

Equations (27)∼(31) are the OFRF based representation for the force transmissibility of Eq. (3) or (8). It can be seen that this representation is an explicit polynomial function of the system's nonlinear characteristic parameters $r_{(2p+1)}$, $(p = 1, 2, \ldots, P)$.

If $r_{(2p+1)} = 0, (p = 1, 2, \cdots, P)$ i.e., there is no nonlinear damping in the system,

$$T_R(\Omega) = |F_{\text{out}}(\Omega)| = \left|(k_1 + jc_1\Omega)\sum_{i=1}^{n}\frac{-\bar{\Phi}_{Ji}\Phi_{1i}}{L_{y_i}(\Omega)}\right|, \tag{32}$$

which is the expression for the force transmissibility of linear MDOF structures.

When an anti-symmetric nonlinear damping is introduced, $r_{(2p+1)} \neq 0, (p = 1, 2, \cdots, P)$, the force transmissibility is different from the result given by Eq. (32), and the difference is a function of both the nonlinear anti-symmetric damping characteristic parameters $r_{(2p+1)}$, $(p = 1, 2, \cdots, P)$ and frequency Ω. In the next section, the effects of parameters $r_{(2p+1)}$, $(p = 1, 2, \cdots, P)$ on the value of $T_R(j\Omega)$ will be analyzed to reveal the significant benefits of a nonlinear anti-symmetric damping characteristic on vibration isolation of MDOF structures.

The effects of nonlinear anti-symmetric damping on force transmissibility of MDOF structures can be described by the two propositions below.

Proposition 1: When the system shown in Fig. 1 works over the non-resonant frequency ranges, i.e., $\Omega << \min(\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)$ or $\Omega >> \max(\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)$, the force transmissibility $T_R(\Omega)$ can be approximated as

$$T_R(\Omega) \approx |\Gamma_1(\Omega)|\,. \tag{33}$$

Proposition 2: when the system shown in Fig. 1 works over the resonant frequency ranges, i.e., $\Omega \approx \bar{\omega}_i$ $(i = 1, 2, \ldots, n)$, there must exist $\bar{\sigma}_{(2\bar{P}+1)} > 0$ $(\bar{P} = 1, 2, \cdots, P)$ such that if $0 < r_{(2\bar{P}+1)} < \bar{\sigma}_{(2\bar{P}+1)}$, then

$$\frac{\partial\,[T_R(\Omega)]^2}{\partial r_{(2\bar{P}+1)}} < 0. \tag{34}$$

The two propositions can be mathematically proved with the OFRF expression of the transmissibility. For the space limit, the details of the proofs are omitted here.

Proposition 1 shows that a nonlinear anti-symmetric damping characteristic has almost no effect on the transmissibility of MDOF structures over the frequency ranges where the frequencies are much lower than the structure's smallest resonant frequency or much higher than the structure's biggest resonant frequency.

Proposition 2 indicates that the system force transmissibility over the resonant frequency ranges can be reduced by increasing the value of any parameter in the system's nonlinear anti-symmetric damping characteristic.

The two propositions together indicate that the MDOF vibration isolator with a nonlinear anti-symmetric damping characteristic has great potential to overcome the limitation with linear vibration isolators. An effective exploitation of this capability of nonlinear vibration isolators could provide a novel passive solution to the dilemma associated with the design of passive viscously damped linear vibration isolators.

In order to verify the significant effects of a nonlinear anti-symmetric damping characteristic on vibration isolation of MDOF structures, numerical simulation studies were conducted by applying the Runge-Kutta method to a 6-DOF structure where a nonlinear damper with anti-symmetric damping characteristic is fitted between the 3rd and 4th masses, i.e., $Q = 4$. The values of system parameters are taken as

$$m_1 = m_2 = \ldots = m_6 = 1,\ k_1 = k_3 = 7\,200,\ k_2 = 0.9k_1,\ k_4 = 0.8k_1,\ k_5 = 1.1k_1,\ k_6 = 0.6k_1,\ \mu = 0.001.$$

For this 6-DOF structure, there are a total 6 normal modes whose associated resonant frequencies can be determined by using the modal analysis technique in Ref. 16, and the results are given in Table 1. The frequency Ω of the sinusoidal external force input over the range of $2 < \Omega < 40$ (Hz) was considered to investigate the effect of an anti-symmetric nonlinear viscous damping characteristic on the force transmissibility over all resonant regions.

Case study 1: $r_3 \neq 0;\ r_p = 0,\ p \neq 3$

In this case study, comparisons were made between the results obtained when the nonlinear damping characteristic parameter r_3 was chosen as $r_3 = 0$, $r_3 = 3.7 \times 10^4$ and $r_3 = 1.5 \times 10^5$, respectively.

Figures 2, 3 show the results obtained in the cases where the external force was applied on the 6th, 4th and 3rd mass, i.e., $J = 6$, 4 and 3, respectively. These results clearly indicate that the introduction of a nonlinear anti-symmetric damping characteristic can not only reduce $T_R(\Omega)$ and suppress vibration over the resonant frequencies but can also keep $T_R(\Omega)$ almost unchanged over the other frequency ranges.

Case study 2: $r_3 \neq 0; r_5 \neq 0; r_p = 0, p \neq 3, 5$

Figure 4 show the results when the parameters r_3 and r_5 were taken as, $r_3 = r_5 = 0$, $r_3 = 5.4 \times 10^4$, $r_5 = 1.9 \times 10^8$, and $r_3 = 5.4 \times 10^4$ and $r_5 = 1.9 \times 10^8$, respectively. It can be seen that the introduction of either term r_3 or r_5 can all effectively reduce the transmissibility $T_R(\Omega)$ over the resonant frequencies; the introduction of both terms r_3 and r_5 can achieve a better effect than when the anti-symmetric damping characteristic is only composed of term r_3 or r_5.

Table 1. The resonant frequencies of the 6-DOF structural system.

	Mode 1	Mode 2	Mode 3	Mode 4	Mode 5	Mode 6
Resonant frequency/Hz	3.14	9.00	13.76	18.51	23.00	25.20

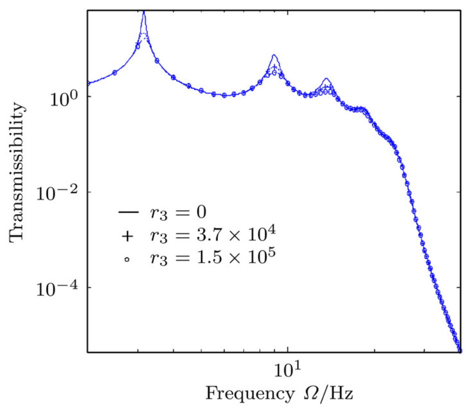

Fig. 2. The force transmissibility of the simulated system under different values of parameter r_3 when the external force was applied to mass 6 ($J = 6$).

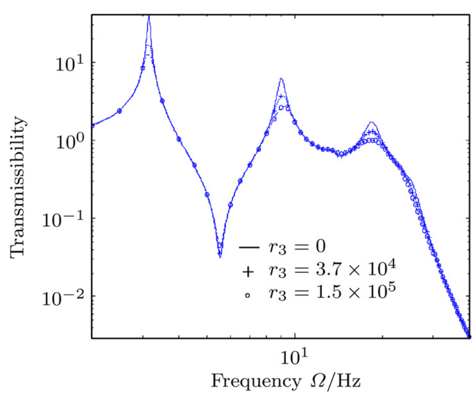

Fig. 3. The force transmissibility of the simulated system under different values of parameter r_3 when the external force was applied to mass 3 ($J = 3$).

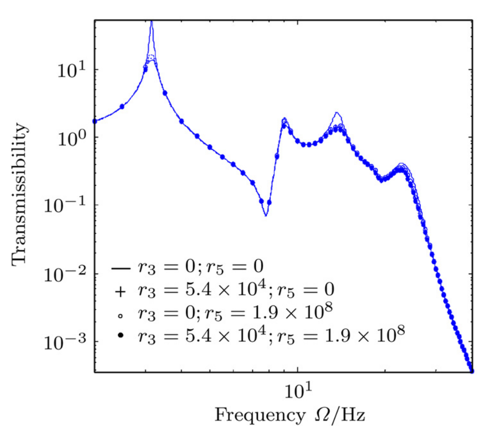

Fig. 4. The force transmissibility of the simulated system under different values of parameters r_3 and r_5 when the external force was applied to mass 4 ($J = 4$).

The simulation studies for the above three cases clearly confirm the theoretical analysis results.

In this study, the force transmissibility of MDOF structures with an anti-symmetric nonlinear viscous damping device is investigated by using Volterra series theory. The results reveal that the introduction of an anti-symmetric nonlinear damping characteristic can significantly reduce the transmissibility over the resonant frequencies of the system while almost not affects the transmissibility over other frequency regions. These extend the conclusions the authors have reached for SDOF structural systems in Ref. 15 to much more general cases, and have significant implication for the design of viscously damped MDOF vibration isolators for a wide range of practical applications.

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1. C. E. Crede, *Vibration and Shock Isolation* (John Wiley, New York, 1951).
2. J. I. Soliman, and E. Ismailzadeh, J. Sound Vib **36**, 527 (1974).
3. J. C. Snowdon, J. Acoust Soc. Am. **66**, 1245 (1979).
4. Y. H. Yu, N. G. Naganathan, and R. V. Dukkipati, Mech. Mach. Theory **36**, 123 (2001).
5. R. A. Ibrahim, J. Sound Vib. **314**, 371 (2008).
6. J. E. Ruzicka, and T. F. Derby, *Influence of Damping in Vibration Isolation* (The Shock and Vibration Information Center, Washington, 1971).
7. Y. K. Dange, and S. R. Gore, Indian J. Techn. **16**, 282 (1978).
8. L. Jiang, D. Stredulinsky, and J. Szabo, et al, Canadian Acoustics - Acoustiqu **30**, 70 (2002).
9. M. S. Hundal, and P. S. Parnes, J. Sound Vib. **64**, 371 (1979).
10. S. M. Metwalli, Optimum nonlinear suspension system. In: *ASME Design Engineering Technical Conference*, (Cincinnati, USA, 1985).
11. X. Yu, S. J. Zhu, and S.Y. Liu, Nonlinear normal modes for multi-degree-of-freedom nonlinear vibration isolation system. In: *Proceedings of the ASME Dynamic Systems and Control Division* (Chicago, USA, 2006).
12. G. N. Jazar, R. Houim, and A. Narimani, et al, J. Sound Vib. **12**, 1205 (2006).
13. Z. Q. Lang, S. A. Billings, and R. Yue, et al, Automatica **143**, 805 (2007).
14. Z. K. Peng, and Z.Q. Lang, J. Sound Vib. **318**, 313 (2008).
15. Z. Q. Lang, X. J. Jing, and S. A. Billings, et al, J. Sound Vib. **323**, 352 (2009).
16. J. He, and Z. Fu, *Modal Analysis* (Heinemann press, Butterworth, 2001).
17. Z. Q. Lang, and S. A. Billings, Int. J. Control **64**, 1049 (1996).
18. S. A. Billings, and J. C. Peyton Jones, Int. J. Control **52**, 863 (1990).
19. W. J. Hsueh, J. Sound Vib. **229**, 793 (2000).
20. A. M. R. Ribeiro, J. M. M. Silva, and N. M. M. Maia, Mech. Syst. Signal. Pr. **14**, 29 (2000).